Special Topics in Algorithmic Game Theory
Lecture 4 – Supplementary Notes

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April 20, 2018

1 Myerson's Optimal Mechanism

Theorem 1 (Theorem 5.2 in [20LAGT])**.** *The expected revenue of every DSIC mechanism (in a Bayesian single-parameter environment) equals its expected virtual welfare. More specifically, if the bidders' values are (independently) drawn from distributions*[1] $F_1, F_2, \dots, F_n$ *and* $(\mathbf{x}, \mathbf{p})$ *is a DSIC mechanism, then*

$$\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right] = \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n} \phi_i(v_i)x_i(\mathbf{v})\right], \tag{1}$$

where $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f(v_i)}$ *is the* virtual valuation *function of bidder* i*.*

Proof. Since bidder values $v_1, \dots, v_n$ are drawn independently, due to the linearity of expectation, it is enough to prove (1) just "per-bidder", i.e. show that

$$\mathbf{E}_{v_i\sim F_i}\left[p_i(\mathbf{v})\right] = \mathbf{E}_{v_i\sim F_i}\left[\phi_i(v_i)x_i(\mathbf{v})\right], \tag{2}$$

for any bidder i and values $\mathbf{v}_{-i}$ of the other players. To see this formally, assume that (2) holds and then observe that indeed:

$$\begin{aligned}
\mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n} p_i(\mathbf{v})\right] &= \sum_{i=1}^{n} \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[p_i(\mathbf{v})\right] \\
&= \sum_{i=1}^{n} \mathbf{E}_{\mathbf{v}_{-i}\sim\mathbf{F}_{-i}}\left[\mathbf{E}_{v_i\sim F_i}\left[p_i(\mathbf{v})\right]\right] \\
&= \sum_{i=1}^{n} \mathbf{E}_{\mathbf{v}_{-i}\sim\mathbf{F}_{-i}}\left[\mathbf{E}_{v_i\sim F_i}\left[\phi_i(v_i)x_i(\mathbf{v})\right]\right] \\
&= \sum_{i=1}^{n} \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\phi_i(v_i)x_i(\mathbf{v})\right] \\
&= \mathbf{E}_{\mathbf{v}\sim\mathbf{F}}\left[\sum_{i=1}^{n} \phi_i(v_i)x_i(\mathbf{v})\right],
\end{aligned}$$

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[1] Recall that we are assuming that distributions F_i are continuous, with strictly positive density functions f_i, supported over a bounded interval $[0, v_i^{\max}]$.

which is exactly (1).

So now we focus on proving (2). Fix player i and values $\mathbf{v}_{-i}$. For simplicity also denote $v \equiv v_i$, $x_i(v) \equiv x_i(v, \mathbf{v}_{-i})$, $p(v) \equiv p_i(v, \mathbf{v}_{-i})$, $F(v) \equiv F_i(v)$ and $\phi(v) \equiv \phi_i(v)$, for all $v \in [0, v_i^{\max}]$. Then:

$$
\begin{aligned}
\mathbf{E}_{v_i \sim F_i}\left[p_i(\mathbf{v})\right] &= \int_0^{v^{\max}} p(v) f(v)\, dv \\
&= \int_0^{v^{\max}} \left(v \cdot x(v) - \int_0^v x(z)\, dz \right) f(v)\, dv, && \text{by Myerson's payment formula}^2, \\
&= \int_0^{v^{\max}} v x(v) f(v)\, dv - \int_0^{v^{\max}} \int_0^v x(z) f(v)\, dz\, dv \\
&= \int_0^{v^{\max}} v x(v) f(v)\, dv - \int_0^{v^{\max}} \int_z^{v^{\max}} x(z) f(v)\, dv\, dz, && \text{changing the order of integration,} \\
&= \int_0^{v^{\max}} v x(v) f(v)\, dv - \int_0^{v^{\max}} x(z) \int_z^{v^{\max}} f(v)\, dv\, dz \\
&= \int_0^{v^{\max}} v x(v) f(v)\, dv - \int_0^{v^{\max}} x(z)(1 - F(z))\, dz \\
&= \int_0^{v^{\max}} v x(v) f(v) - x(v)(1 - F(v))\, dv \\
&= \int_0^{v^{\max}} x(v) \left[v - \frac{1 - F(v)}{f(v)} \right] f(v)\, dv \\
&= \int_0^{v^{\max}} x(v) \phi(v) f(v)\, dv \\
&= \mathbf{E}_{v_i \sim F_i}\left[x_i(\mathbf{v}) \phi(\mathbf{v}) \right].
\end{aligned}
$$

□

It is important to emphasize here that, although in the above proof we assume that distributions F_i are supported in intervals of the form $[0, v_i^{\max}]$, this is merely for technical convenience; it is a matter of straightforward calculus to adapt the proof for distributions having positive densities over general nonnegative intervals, i.e., of the form $[\alpha_i, \beta_i]$ with $\alpha_i \geq 0$ and β_i possibly being ∞. So, feel free to use the following whenever designing optimal auctions, and see also the detailed example in Section 1.1 below for a demonstration:

Remark 1. *Theorem 1 holds, exactly* as-it-is, *even if the supports of distributions $F_1, \ldots, F_n$ are arbitrary intervals within the nonnegative reals.*[3]

Discrete Distributions. Theorem 1 can be further generalized to incorporate way more general distributions, and in particular non-continuous ones, that may possess point masses. However, doing this formally is a non-trivial task, beyond the scope of the current course (see, e.g., Skreta [4], Monteiro and Svaiter [3]). Nevertheless, we just want to briefly mention a couple of interesting points, for the interested reader. First, one elegant way of handling the point masses is by resorting to Dirac delta functions [4].

Secondly, for the special case of discrete distributions, one can adapt the definition of the virtual valuation function ϕ_i, in a natural way, in order to get Theorem 1 working, as expected after translating it to the realm of discrete supports. For example, we briefly mention that, if F_i is a distribution over $z_i^1, \ldots, z_i^K$, with point z_i^k having probability mass f_i^k, for all $k = 1, \ldots, K$, then we can define the virtual valuation as

$$\phi_i^k = z_i^k - (z_i^{k+1} - z_k) \frac{1 - F_i^k}{f_i^k},$$

where $F_i^k = \sum_{j=1}^i f_i^k$ is the cdf on z_i^k. For more details on this, see the original papers of Elkind [2] and Bergemann and Pesendorfer [1].

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[2] See Eq. (1) in the Supplementary Notes for Lecture 3.

[3] We still require, though, that the distributions have finite expectation.

1.1 An Example

Consider a single-item, Bayesian auction environment, with two bidders. The valuation of the first bidder is drawn from an exponential distribution with parameter 1 , i.e. $F_1(x) = 1 - e^{-x}$ for $x \in [0, \infty)$; the valuation of the second follows a uniform distribution over $[1, 4]$, i.e. $F_2(y) = \frac{y-1}{3}$ for $y \in [1, 4]$. Since we are in the 2-dimensional Euclidean space, for simplicity we will just use coordinates x and y, respectively, to denote that valuations v_1, v_2 of the two bidders.

First, we compute the virtual valuations:

$$\phi_1(x) = x - \frac{1 - F_1(x)}{F_1'(x)} = x - \frac{e^{-x}}{e^{-x}} = x - 1$$

and

$$\phi_2(y) = y - \frac{1 - F_2(y)}{F_2'(y)} = y - \frac{\frac{4}{3} - \frac{y}{3}}{\frac{1}{3}} = 2y - 4.$$

These are both (strictly) increasing functions, and thus our distributions satisfy the desired regularity condition. Next, by solving

$$\phi_1(x) \geq 0 \iff x - 1 \geq 0 \iff x \geq 0$$

$$\phi_2(y) \geq 0 \iff 2y - 4 \geq 0 \iff y \geq 2$$

and

$$\phi_1(x) \leq \phi_2(y) \iff x - 1 \leq 2y - 4 \iff x \leq 2y - 3 \iff y \geq \frac{x}{2} + \frac{3}{2}, \tag{3}$$

the bidding space $[0, \infty] \times [1, 4]$ gets partitioned in the following critical regions (see Figure 1):

$$\begin{aligned}
A &= \{(x, y) \mid x > 1 \;\; \wedge \;\; 1 < y < 2\} \\
B &= \{(x, y) \mid x > 2y - 3 \;\; \wedge \;\; 2 < y < 4\} \\
C &= \left\{(x, y) \mid x > 1 \;\; \wedge \;\; \frac{x}{2} + \frac{3}{2} < y < 4\right\} \\
D &= \{(x, y) \mid 0 < x < 1 \;\; \wedge \;\; 2 < y < 4\} \\
N &= \{(x, y) \mid 0 < x < 1 \;\; \wedge \;\; 1 < y < 2\}
\end{aligned}$$

In region A, the item gets allocated to bidder 1, since she is the only one with a positive virtual valuation, for a payment of $p_1^A = p_1^A(x, y) = 1$, her monopoly reserve $\phi_1^{-1}(0)$. Similarly, in region D, the item gets allocated to bidder 2 for a price of $p_2^D = \phi_2^{-1}(0) = 2$. In the remaining regions, namely B and C, both players have positive virtual valuations, so we allocate to the highest of them. More specifically, in region B we sell to bidder 1 and in region C to bidder 2. The payments are determined by the "critical" value that the winning bidder needs to bid in order to still keep the item, given essentially by (3). Formally, for example in region B, player 1 has to pay

$$p_1^B = \phi_1^{-1}(\phi_2(y)) = \phi_1^{-1}(2y - 4) = (2y - 4) + 1 = 2y - 3,$$

which, equivalently, is exactly the smallest x such that $\phi_1(x) \geq \phi_2(y)$ (see (3)). For the same reasons, $p_2^C = \frac{1}{2}x + \frac{3}{2}$.

Finally, observe that there is region N where, although inside the feasible bidding space $[0, \infty) \times [1, 4]$, the item is not sold to any bidder: they both have negative virtual valuations.

Notice how we use *strict* inequalities when defining our critical regions A, B, C and D; this is to avoid dealing with ties. For example, exactly on the boundary between B and C, which lies on the line $y = \frac{1}{2}x + \frac{3}{2}$, we can allocate the item arbitrarily to one of the bidders, without this affecting the expected revenue of our auction — this is a set of zero Lebesgue measure in $\mathbb{R}^2$.

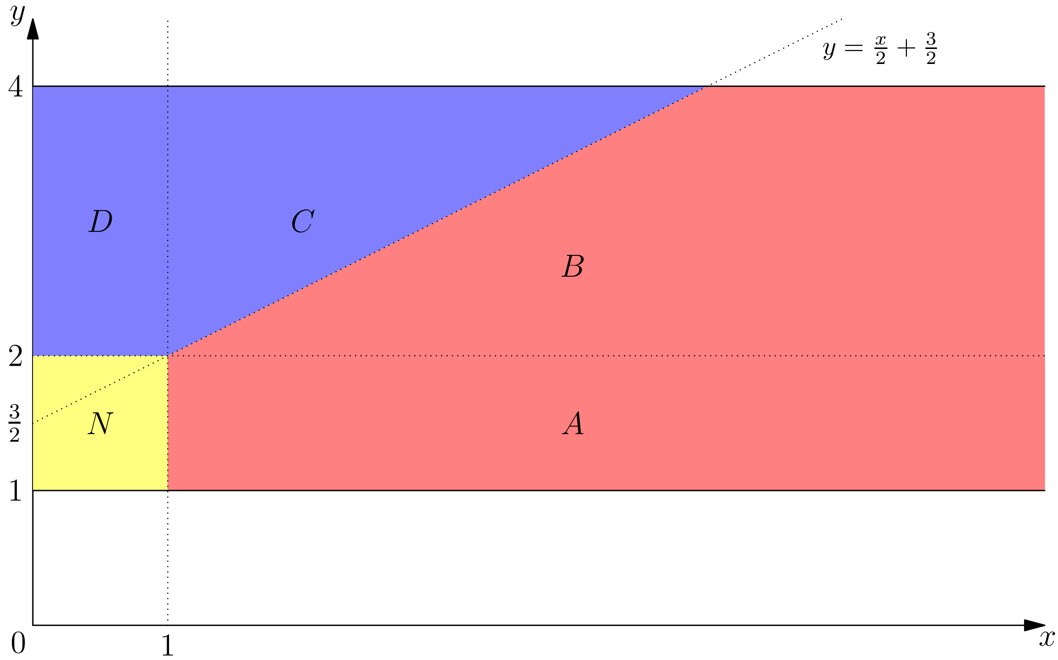

Figure 1: Selling a single item to two bidders with valuations drawn from an exponential distribution and a uniform distribution over $[1, 4]$, respectively. The bidding space $[0, \infty) \times [1, 4]$ is partitioned according the corresponding allocation rule of the *optimal* (revenue-maximizing) auction. Bidder 1 wins the item in regions $A \cup B$ (red) and bidder 2 in $C \cup D$ (blue). The item remains unsold in region N (yellow).

References

[1] D. Bergemann and M. Pesendorfer. Information structures in optimal auctions. *Journal of Economic Theory*, 137(1):580–609, nov 2007. doi: 10.1016/j.jet.2007.02.001.

[2] E. Elkind. Designing and learning optimal finite support auctions. In *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '07, pages 736–745, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics. ISBN 978-0-898716-24-5. URL http://dl.acm.org/citation.cfm?id=1283383.1283462.

[3] P. K. Monteiro and B. F. Svaiter. Optimal auction with a general distribution: Virtual valuation without densities. *Journal of Mathematical Economics*, 46(1):21–31, jan 2010. doi: 10.1016/j.jmateco.2009.06.004.

[4] V. Skreta. Optimal auctions with general distribution. Technical report, New York University, Leonard N. Stern School of Business, Department of Economics, 2008.