Technische Universität München, Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

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Exercise Sheet 1 - Due Wednesday, April 18

Bonus system: *Students are expected to solve these exercise sets, and they can apply for a grade bonus by writing down and handing in their answers, either in person or by e-mail to* `diogo.pocas@tum.de`*. Each student that hands in a sufficient amount of satisfactory solutions achieves a grade bonus, which improves the grade of a passed exam by one grade (e.g. a 4.0 becomes a 3.7, or a 3.7 becomes a 3.3, etc. Note that a 1.0 cannot be improved).* ***No late submissions accepted.***

Exercise 1.1 (Olympic badminton tournament, Exercise 1.1 from [20LAGT])
Give at least two suggestions for how to modify the Olympic badminton tournament format to reduce or eliminate the incentive for a team to intentionally lose a match.

Exercise 1.2 (*A Beautiful Mind*, Exercise 1.2 from [20LAGT])
Watch the scene[1] from the movie *"A Beautiful Mind"* (2001, starring Russell Crowe) that purports to explain what a Nash equilibrium is. The scenario described is most easily modeled as a game with four players (the men), each with the same five actions (the women). Explain why the solution proposed by the John Nash character is actually *not* a Nash equilibrium.

Exercise 1.3 (Equivalent definition of mixed Nash equilibrium, related to Exercise 13.1 from [20LAGT])
Prove that the inequality condition in the definition of a *mixed* Nash equilibrium[2] can be equivalently replaced by the following one:

$$\mathbf{E}_{s\sim\sigma}\left[u_i(\mathbf{s})\right] \geq \mathbf{E}_{s_i'\sim\sigma_i',\mathbf{s}_{-i}\sim\sigma_{-i}}\left[u_i(s_i',\mathbf{s}_{-i})\right],$$

for all mixed strategies σ_i' of player i. In other words, considering just pure-strategy deviation in the original definition is without loss.

Exercise 1.4 (Single-item third-price auction, Exercise 2.1 from [20LAGT])
Consider a single-item auction with at least three bidders. Prove that selling the item to the highest bidder, at a price equal to the third-highest bid, yields an auction that is *not* DSIC.

Exercise 1.5 (Second-price auction for identical copies, Exercise 2.3 from [20LAGT])
Suppose there are k identical copies of an item and $n > k$ bidders. Suppose also that each bidder can receive at most one item. What is the analog of the second-price auction? Prove that your auction is DSIC.

Exercise 1.6 (Maximum welfare for sponsored search auctions, Exercise 2.8 from [20LAGT])
Recall the sponsored search setting of Section 2.6 of [20LAGT], in which bidder i has a valuation v_i per click. There are k slots with click-through rates (CTRs) $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$. The social welfare of an assignment of bidders to slots is $\sum_{i=1}^{n} v_i x_i$, where x_i equals the CTR of the slot to which bidder i is assigned (or 0 if i is not assigned to any slot).

[1] It's easy to find on YouTube, e.g. search for "Beautiful Mind bar scene".
[2] See Equation (3) in the Supplementary Notes provided after Lecture 1 (Wednesday, April 11, 2018).

Please turn over.

Prove that the social welfare is maximized by assigning the bidder with the i-th highest valuation to the i-th best slot for $i = 1, 2, \ldots, k$.

Exercise 1.7 (Online single-item auctions, Problem 2.1 from [20LAGT])
This problem considers *online* single-item auctions, where bidders arrive one-by-one. Assume that the number n of bidders is known, and that bidder i has a private valuation v_i. We consider auctions of the following form.

Online Single-Item Auction

For each bidder arrival $i = 1, 2, \ldots, n$:

if the item has not been sold in a previous iteration, formulate a price p_i and then accept a bid b_i from bidder i

if $p_i \leq b_i$, then the item is sold to bidder i for a price of p_i; otherwise, bidder i departs and the item remains unsold

(a) Prove that an auction of this form is DSIC.

(b) Assume that bidders bid truthfully. Prove that if the valuations of the bidders and the order in which they arrive are arbitrary, then for every constant $c > 0$ independent of n, there is *no* deterministic online auction that always achieves social welfare at least c times the highest valuation.

(c) Assume that bidders bid truthfully. Prove that there is a constant $c > 0$, independent of n, and a deterministic online auction with the following guarantee: for every unordered set of n bidder valuations, if the bidders arrive in a uniformly random order, then the expected welfare of the auction's outcome is at least c times the highest valuation.

This problem set will be discussed in the tutorials on April 20th, 2018.