Technische Universität München, Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

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Exercise Sheet 6 - Due Wednesday, May 23

Hand in your answers, either in person or by e-mail to `diogo.pocas@tum.de` *(preferably as a single file).* ***No late submissions accepted.***

Exercise 6.1 (Pure strategies in the support of a mixed Nash equilibrium)
Show that, at any mixed Nash equilibrium, all pure strategies in the support of any player induce the same expected cost. Formally, if $\sigma = \sigma_1 \times \ldots \times \sigma_n$ is a mixed Nash equilibrium and σ_i assigns strictly positive probabilities to both (pure) strategies s_i, s_i', then

$$\mathbf{E}_{\mathbf{s}_{-i} \sim \sigma_{-i}} \left[C_i(s_i, \mathbf{s}_{-i}) \right] = \mathbf{E}_{\mathbf{s}_{-i} \sim \sigma_{-i}} \left[C_i(s_i', \mathbf{s}_{-i}) \right].$$

Exercise 6.2 (Equilibrium concepts for the traffic light game)
Recall the traffic light game seen in class with two agents, each having one of two possible actions ('Stop' or 'Go').[1] We have seen that

- ('Stop','Go') and ('Go','Stop') are pure Nash equilibria;
- there is a mixed Nash equilibrium where each agent plays 'Go' with probability $\frac{1}{5}$;
- there is a correlated equilibrium which randomizes uniformly between the two pure Nash equilibria.

(a) Are there any other mixed Nash equilibria? (prove or give an example)

(b) Are there any coarse correlated equilibria which are *not* correlated equilibria? (prove or give an example)

(c) Show that the proper[2] mixed Nash equilibrium given in class achieves the *worst* expected social cost among all coarse correlated equilibria.

(*Hint: feel free to use any LP solver you wish, in any programming language you like, such as Mathematica, Matlab, Python,* `http://www.phpsimplex.com`, *etc.*)

Exercise 6.3 (Price of Anarchy of coarse correlated equilibria for congestion games, related to Theorem 14.4 from [20LAGT])
Prove that the Price of Anarchy of coarse correlated equilibria for congestion games with affine cost functions is upper bounded by $\frac{5}{2}$.

Exercise 6.4 (Best-response dynamics cycles forever, Exercise 16.1 from [20LAGT])
Exhibit a game with a pure Nash equilibrium and an initial outcome from which best-response dynamics cycles forever.

Exercise 6.5 (Price of Anarchy of mixed Nash equilibria for scheduling games, Problem 13.1 in [20LAGT])

[1] Its cost matrix can be found in page 177 of your textbook [20LAGT].
[2] That is, the mixed Nash equilibrium which is *not* pure.

Please turn over.

Recall the class of cost-minimization games introduced in Exercise 5.3 from last week (also Problem 12.3 in [20LAGT]), where each agent $i = 1, 2, \dots, k$ has a positive weight w_i and chooses one of m identical machines to minimize her load. We again consider the objective of minimizing the makespan, defined as the maximum load of a machine. Prove that, as k and m tend to infinity, the worst-case Price of Anarchy of *mixed* Nash equilibria in such games is *not* upper bounded by any constant.

(*Hint: you can invoke well-known properties of 'balls-into-bins' occupancy problems.*)

Exercise 6.6 (Best-response dynamics for scheduling games, Problem 16.1 from [20LAGT])
Recall the class of cost-minimization games introduced in Exercise 5.3 from last week (also Problem 12.3 in [20LAGT]), where each agent $i = 1, 2, \dots, k$ has a positive weight w_i and chooses one of m identical machines to minimize her load. Consider the following restriction of best-response dynamics:

Maximum-Weight Best-Response Dynamics

While the current outcome $\mathbf{s}$ is not a pure Nash equilibrium:

> among all agents with a beneficial deviation, let i denote an agent with the largest weight w_i and s_i' a best response to $\mathbf{s}_{-i}$
>
> update the outcome to $(s_i', \mathbf{s}_{-i})$

Prove that maximum-weight best-response dynamics converges to a pure Nash equilibrium after at most k iterations.

This problem set will be discussed in the tutorials on May 25th, 2018.