Technische Universität München,
Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

Dr. Yiannis Giannakopoulos | Dr. Diogo Poças

Exercise Sheet 2 - Due Wednesday, April 25

Exercise 2.1 (Sponsored search with quality, Exercise 3.4 from [20LAGT])
Consider the following extension of the sponsored search setting described in class. Each bidder i now has a publicly known *quality* β_i, in addition to a private valuation v_i per click. As usual, each slot j has a CTR α_j, and $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_k$. We assume that if bidder i is placed in slot j, then the probability of a click is $\beta_i \alpha_j$. Thus bidder i derives value $v_i \beta_i \alpha_j$ from the jth slot.

Describe the welfare-maximizing allocation rule in this generalized sponsored search setting. Prove that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

Exercise 2.2 (Revenue target auctions, Problem 3.2 from [20LAGT])
This problem considers a k-unit auction (Exercise 1.5 from last week) in which the seller has a specific revenue target R. Consider the following algorithm that, given bids $\mathbf{b}$ as input, determines the winning bidders and their payments.

Revenue Target Auction

initialize a set S to the top k bidders

while there is a bidder $i \in S$ with $b_i < R/|S|$ **do**

remove an arbitrary such bidder from S

allocate an item to each bidder of S (if any), and charge each of them a price equal to the maximum between $R/|S|$ and the $(k+1)$-th highest bid.

(a) Give an explicit description of the allocation rule of the Revenue Target Auction, and prove that it is monotone.

(b) Conclude from Myerson's lemma that the Revenue Target Auction is a DSIC mechanism.

(c) Prove that whenever the DSIC and welfare-maximizing k-unit auction (Exercise 1.5 from last week) obtains revenue at least R, the Revenue Target Auction obtains revenue R.

(d) Prove that there exists a valuation profile for which the Revenue Target Auction obtains revenue R but the DSIC and welfare-maximizing auction earns revenue less than R.

Exercise 2.3 (Single-item auction with two uniform bidders, Exercise 5.1 from [20LAGT])
Consider a single-item auction with two bidders with valuations drawn independently from the uniform distribution on $[0, 1]$.

(a) Prove that the expected revenue obtained by a second-price auction (with no reserve) is $\frac{1}{3}$.

Please turn over.

(b) Prove that the expected revenue obtained by a second-price auction with reserve $\frac{1}{2}$ is $\frac{5}{12}$.

Exercise 2.4 (Virtual valuations, Exercises 5.2 and 5.3 from [20LAGT])
Compute the virtual valuation function of the following distributions. Which of these distributions are regular?

(a) The uniform distribution on $[0, a]$ with $a > 0$.

(b) The exponential distribution with rate $\lambda > 0$ (on $[0, \infty)$).

(c) The distribution given by $F(v) = 1 - \frac{1}{(v+1)^c}$ on $[0, \infty)$, where $c > 0$ is some constant.

Exercise 2.5 (Optimal k-unit auction, Exercise 5.7 from [20LAGT])
Consider a k-unit auction (Exercise 1.5 from last week) in which bidders' valuations are drawn i.i.d. from a regular distribution F. Describe an optimal auction. Which of the following does the reserve price depend on: k, n, or F?

Exercise 2.6 (Bayes-Nash equilibrium and first-price auctions, Problem 5.3 from [20LAGT])
This problem introduces the Bayes-Nash equilibrium concept and compares the expected revenue earned by first-price and second-price single-item auctions.

First-price auctions have no dominant strategies, and we require a new concept to reason about them. For this problem, assume that bidders' valuations are drawn i.i.d. from a commonly known distribution F. A *strategy* of a bidder i in a first-price auction is a predetermined plan for bidding — a function $b_i(\cdot)$ that maps her valuation v_i to a bid $b_i(v_i)$. The semantics are: "when my valuation is v_i, I will bid $b_i(v_i)$." We assume that bidders' strategies are common knowledge, with bidders' valuations (and hence induced bids) private as usual.

A strategy profile $b_1(\cdot), \ldots, b_n(\cdot)$ is a *Bayes-Nash equilibrium* if every bidder always bids optimally given her information — if for every bidder i and every valuation v_i, the bid $b_i(v_i)$ maximizes i's expected utility, where the expectation is with respect to the distribution over others bids induced by F and $\mathbf{b}_{-i}$.

(a) Suppose F is the uniform distribution on $[0, 1]$. Verify that setting $b_i(v_i) = v_i(n-1)/n$ for every i and v_i yields a Bayes-Nash equilibrium.

(b) Prove that the expected revenue of the seller (over $\mathbf{v}$) at this equilibrium of the first-price auction is exactly the expected revenue of the seller in the truthful outcome of a second-price auction.

(c) Extend the conclusion in (b) to every continuous and strictly increasing distribution function F on $[0, 1]$.

This problem set will be discussed in the tutorials on April 27th, 2018.