Technische Universität München, Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

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Exercise Sheet 5 - Due Wednesday, May 16

Hand in your answers, either in person or by e-mail to `diogo.pocas@tum.de` *(preferably as a single file).* ***No late submissions accepted.***

Exercise 5.1 (Two-hop path Nash equilibrium, Exercise 12.5 from [20LAGT])
Recall the four-agent atomic selfish routing network in Figure 12.3 from [20LAGT]. Verify that if each agent routes her traffic on her two-hop path, then the result is an equilibrium flow.

Exercise 5.2 (Price of Anarchy with two agents, Problem 12.2(b) from [20LAGT])
Recall the four-agent atomic selfish routing network in Figure 12.3 from [20LAGT], where the Price of Anarchy is 2.5. How large can the Price of Anarchy be with affine cost functions and only *two* agents?

Exercise 5.3 (Price of Anarchy of scheduling games, Problem 12.3 from [20LAGT])
This problem studies a scenario with k agents, where agent i has a positive weight w_i. There are m identical machines. Each agent chooses a machine, and wants to minimize the *load* of her machine, defined as the sum of the weights of the agents who choose it. This problem considers the objective of minimizing the *makespan*, defined as the maximum load of a machine. A *pure Nash equilibrium* is an assignment of agents to machines so that no agent can unilaterally switch machines and decrease the load she experiences.

(a) Prove that the makespan of a pure Nash equilibrium is at most twice that of the minimum possible.

(b) Prove that, as k and m tend to infinity, pure Nash equilibria can have makespan arbitrarily close to twice the minimum possible.

Exercise 5.4 (Lower bound on Price of Stability)
Show that the Price of Stability of the class of atomic selfish routing networks with cost functions from the family $\mathcal{C}_d$ of polynomials with maximum degree d (and nonnegative coefficients) is $\Omega(\frac{d}{\log d})$, i.e. at least the value of the Pigou bound $\alpha(\mathcal{C}_d)$.

Exercise 5.5 (Relating potential and cost functions, related to Exercise 13.4 and Problem 15.3 from [20LAGT])
Consider the class of congestion games with cost functions from the family $\mathcal{C}_d$ of polynomials with maximum degree d (and nonnegative coefficients). Prove that, for any strategy profile $\mathbf{s}$,

$$\frac{1}{d+1}C(\mathbf{s}) \leq \Phi(\mathbf{s}) \leq C(\mathbf{s}),$$

where $C(\mathbf{s})$ is the social cost and $\Phi(\mathbf{s})$ is Rosenthal's potential.[1]

Exercise 5.6 (Weighted congestion games with affine cost functions have pure equilibria, related to Exercise 13.6 from [20LAGT])

[1]See Equations (4) and (5), respectively, from Lecture's 8 Supplementary Notes.

Please turn over.

We saw in class[2] that, in general, weighted congestion games may not have pure Nash equilibria. However, for the special case when all resources have affine cost functions, they actually do have. Prove this, by using the following function as a potential:

$$\Phi(\mathbf{s}) = \sum_{e \in E} \left(c_e(f_e(\mathbf{s})) f_e(\mathbf{s}) + \sum_{i : e \in s_i} c_e(w_i) w_i \right).$$

[2]See Section 4 from Lecture's 8 Supplementary Notes.

This problem set will be discussed in the tutorials on May 18th, 2018.