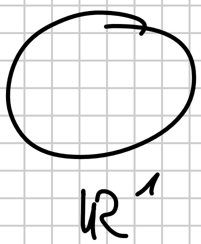
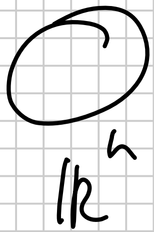
HM2 MW GW Blatt 11





Skalarfeld $\mathbb{R}^n \to \mathbb{R}$

∇f , H_f

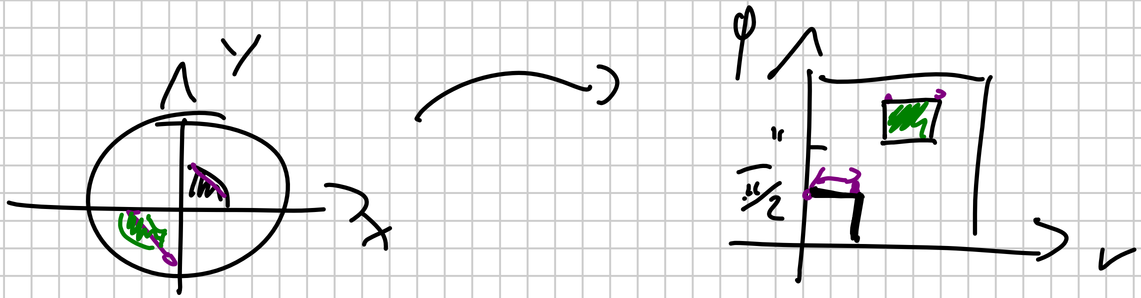
Vektorfeld $\mathbb{R}^n \to \mathbb{R}^m$

$$J_f = \begin{pmatrix} \nabla f_1^T \\ \nabla f_m^T \end{pmatrix}$$

i) EV, EW $\longrightarrow$ Trafo Entkoppeln

ii) Satz impl Fkt Reduktion

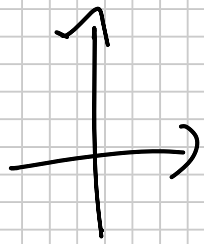
ad i)

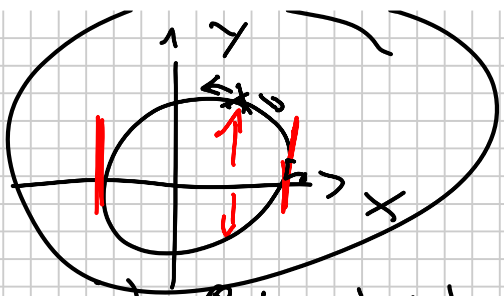


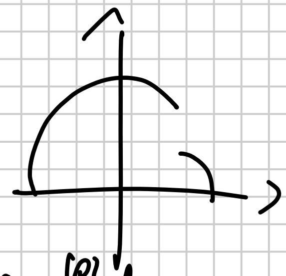
1)

a) $x_1 + x_2 - \sin(x_3) = 0$

$\exp(x_3) - x_1 - x_2^3 = 1$







c) Bed mind. einen Wert

i) stetig diff'bar kat.

iii) $f_y \neq 0$ $\hat{=}$ J_f muss inv.

Bemerkung: $\det A = \prod \lambda_i$

$\text{spur } A = \sum a_{ii} = \sum \lambda_i$

$$J_f = \begin{pmatrix} 1 & 1 & -\cos(x_3) \\ -1 & -3x_2^2 & \exp(x_3) \end{pmatrix}$$

$\det J_f \neq 0$

$A\vec{x} = ?$ ↯ wirklich

$$J_f(0,0,0) = \begin{array}{c} f_1 \\ f_2 \end{array} \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

Einschub Taylor

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots$

Skalar: $f(\vec{x}) = f(\vec{x_0}) + Df(\vec{x_0})^T(\vec{x} - \vec{x_0}) + (\vec{x}-\vec{x_0})^T \frac{H_f}{2!} (\vec{x} - \vec{x_0})$

Vektor: $\vec{f}(\vec{x}) = f(\vec{x_0}) + J(\vec{x_0})(\vec{x} - \vec{x_0}) + \dots$

(x_1, x_2)

$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\det = 1 \cdot 0 - (-1) \cdot 1 = 1$ auflösbar

(x_2, x_3)

$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, $\det = 1$ auflösbar

(x_1, x_3)

$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $\det = 0$ nicht auflösbar

b) $x_1 + x_2 - \sin x_3 = 0$

$\exp x_1 - x_1^2 + x_2 = 1$ $\quad \| \quad$!: „0 auf eine Seite"

i) $\vec{0}$ erfüllt die Gleichungen

ii) die Fkt sind stetig diff'bar

Was ist f? $\mathbb{R}^3 \to \mathbb{R}^2$

$$f = \begin{pmatrix} x_1 + x_2 - \sin x_3 \\ \exp x_1 - x_1^2 + x_2 - 1 \end{pmatrix}$$

iii)

$$J_f = \begin{pmatrix} 1 & 1 & -\cos x_3 \\ -2x_1 + \exp x_1 & 1 & 0 \end{pmatrix}$$

$$J_f(0,0,0) = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$

(x_1, x_2) $\quad \det \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}} = 0$

$\Rightarrow$ nicht auflösbar

(x_2, x_3) $\quad \det \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}} = 1$

auflösbar

(x_1, x_3) $\quad \det \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}} = 1$

auflösbar

Transformation eines Vektorfeldes

$$\vec{x} = \phi(\vec{y})$$

alte Koord. $\quad$ neue Koord.

Trafo-Matrix $T_\phi(\vec{y})$

$\vec{v}_{alt}(\vec{x})$ Vektorfeld

$$\vec{v}_{neu}(\vec{y}) = T_\phi(\vec{y})\, \vec{v}_{alt}(\vec{x})$$
$$= \vec{v}_{alt}(\phi(\vec{y}))$$

Bemerkung

grad f : $\nabla \cdot f = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$

Produkt

hier Skalarprodukt

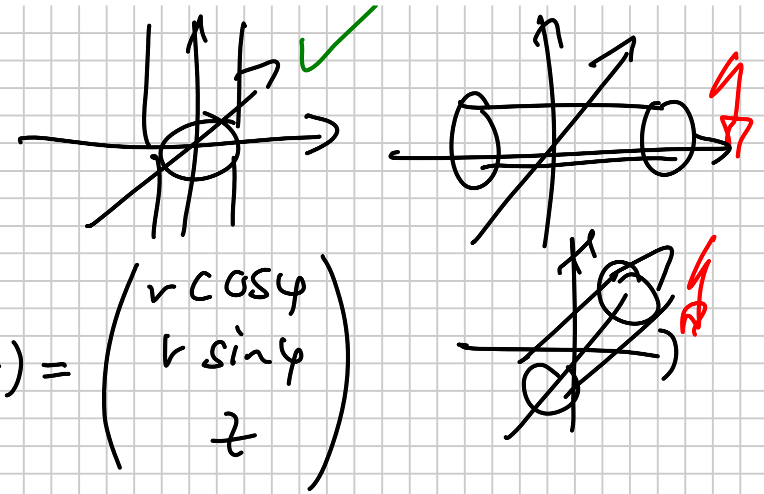
$$\text{div}\, \vec{v} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \vec{v} = \nabla \cdot \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

∇

$$\text{rot}\, \vec{v} = \nabla \times \vec{v} = \begin{pmatrix} \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \\ \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{pmatrix}$$

Aufgabe 2:

$$\vec{v}_{kart}(x,y,z) = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} xz \\ yz \\ \sqrt{x^2+y^2} \end{pmatrix}$$



Zylinderkoord.:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \phi(r,\varphi,z) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \\ z \end{pmatrix}$$

$$\vec{v}_{zyl\,koord} = \underbrace{T_{zyl}}\cdot\underbrace{\vec{v}_{kart}(\phi(r,\varphi,z))}$$

$$= \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{r} \begin{pmatrix} r\cos(\varphi)z \\ r\sin(\varphi)z \\ r \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ 1 \end{pmatrix}$$

~~Einsetz~~

$$\text{rot}\,\vec{v} = \qquad = \frac{1}{\sqrt{x^2+y^2}} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

$$\text{div}\,\vec{v}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \left(\frac{\partial \frac{xz}{\sqrt{x^2+y^2}}}{\partial x}\right) + \left(\frac{\partial \frac{yz}{\sqrt{x^2+y^2}}}{\partial y}\right) + \frac{\partial 1}{\partial z} =$$

$$= \left(\frac{z}{\sqrt{x^2+y^2}} + x \cdot \frac{\left(-\frac{1}{2}\right) \cdot 2x \cdot z}{(x^2+y^2)^{3/2}} \right) + \text{''} + 0$$

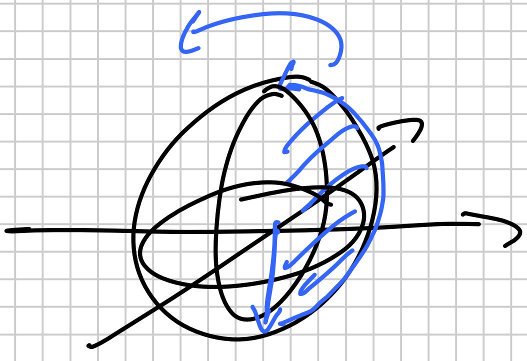
$$= \frac{z\,(x^2+y^2) - x^2 z}{(x^2+y^2)^{3/2}} + \text{''}$$

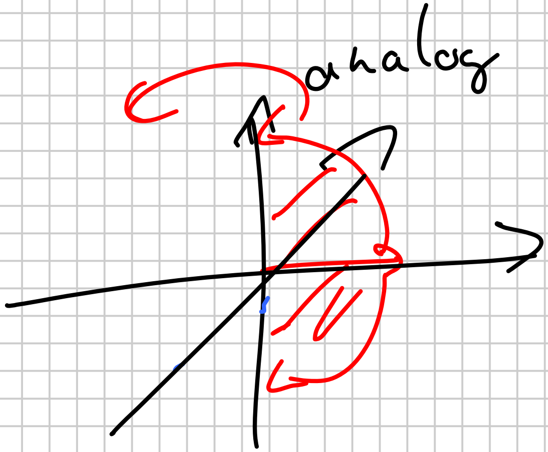
$$= \frac{y^2 z}{(x^2+y^2)^{3/2}} + \frac{x^2 z}{(x^2+y^2)^{3/2}}$$

$$\operatorname{rot} \vec{v}_{zyl.}(r, \varphi, z) = \boxed{\begin{pmatrix} \frac{1}{r}\frac{\partial v_3}{\partial \varphi} - \frac{\partial v_2}{\partial z} \\ \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial r} \\ \frac{1}{r}\frac{\partial (r \cdot v_2)}{\partial r} - \frac{1}{r}\frac{\partial v_1}{\partial \varphi} \end{pmatrix}} = \dots = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\operatorname{div} \vec{v}_{zyl}(r, \varphi, z) = \boxed{\frac{1}{r}\frac{\partial (r v_1)}{\partial r} + \frac{1}{r}\frac{\partial v_2}{\partial \varphi} + \frac{\partial v_3}{\partial z}} = \dots = \frac{z}{r}$$

Aufgabe 3: Kugelkoord





Formeln:

$\mathrm{rot}_{\text{Kugel}}$

$\mathrm{div}_{\text{Kugel}}$

Aufgabe 4:

$$\mathrm{rot}(\mathrm{rot}\,\vec{a}) = -\Delta\vec{a} + \mathrm{grad}(\mathrm{div}\,\vec{a})$$

$$\operatorname{rot} \vec{a} = \begin{pmatrix} \partial_2 a_3 - \partial_3 a_2 \\ \partial_3 a_1 - \partial_1 a_3 \\ \partial_1 a_2 - \partial_2 a_1 \end{pmatrix}$$

$\vdots$

✓

Für Ursprung einer Quadrik:

siehe Skript S. 198/199