Technische Universität München, Department of Mathematics
Chair of Operations Research

Special Topics in Algorithmic Game Theory (MA5226)

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Exercise Sheet 3 - Due Wednesday, May 2

Hand in your answers either in person (by the end of Wednesday's class), or by e-mail to `diogo.pocas@tum.de` *(preferably as a single file).* ***No late submissions accepted.***

Exercise 3.1 (Optimal single-item auction with independent bidders, Exercise 6.1 from [20LAGT])
Consider an n-bidder single-item auction, with bidders' valuations drawn independently from regular distributions $F_1, \ldots, F_n$.

(a) Give a formula for the winner's payment in an optimal auction, in terms of the bidders' virtual valuation functions.

(b) Show by example that, in an optimal auction, the highest bidder need not win, even if they have a positive virtual valuation.

(c) Give an intuitive explanation of why the property in (b) might be beneficial to the expected revenue of an auction.

Exercise 3.2 (Second-price auction with bidder-specific reserve prices, Exercise 6.3 from [20LAGT])
Prove that with regular valuation distributions $F_1, \ldots F_n$, the allocation rule of a second-price auction with bidder-specific reserve prices $r_i = \varphi_1^{-1}(t)$ is monotone.

Exercise 3.3 (Comparing expected revenues with regular bidders, Exercise 6.4 from [20LAGT])
Consider an n-bidder single-item auction, with bidders' valuations drawn i.i.d. from a regular distribution F. Prove that the expected revenue of a second-price auction (with no reserve price) is at least $\frac{n-1}{n}$ times that of an optimal auction.

Exercise 3.4 (Improving the prophet inequality, related to Problem 6.1 from [20LAGT])
This problem investigates improvements to the prophet inequality (Theorem 6.1 from [20LAGT]).

(a) Show that the factor of $\frac{1}{2}$ in the prophet inequality cannot be improved, even for $n = 2$: for every constant $c > \frac{1}{2}$, there are distributions[1] G_1, G_2 such that *every* strategy, threshold or otherwise, has expected value less than $c \cdot \mathbf{E}_{\pi \sim \mathbf{G}} [\max_i \pi_i]$.

(b) Can the factor of $\frac{1}{2}$ in the prophet inequality be improved for the special case of i.i.d. distributions, with $G_1 = \cdots = G_n$?

Exercise 3.5 (Regular distributions: concavity and median pricing, related to Problems 5.1 and 5.2 from [20LAGT])
This problem derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density function f on the interval $[0, v_{\max}]$, with $v_{\max} < +\infty$.

[1] Recall that the prophet inequality does *not* necessarily assume continuity of the probability distributions. Thus, if you like, feel free to use *discrete* distributions in this exercise.

Please turn over.

For a single bidder with valuation drawn from F, for $q \in [0,1]$, define $V(q) = F^{-1}(1-q)$ as the (unique) posted price that yields a probability q of a sale. Define $R(q) = q \cdot V(q)$ as the expected revenue obtained from a single bidder when the probability of a sale is q. The function $R(q)$, for $q \in [0,1]$, is the *revenue curve* of F. Note that $R(0) = R(1) = 0$.

(a) What is the revenue curve for the uniform distribution on $[0,1]$?

(b) Prove that the revenue curve of a regular distribution is concave.

(c) Fix now a single-item, single-bidder environment with valuation drawn from a regular distribution F. Prove that offering to the bidder a price equal to the median of F (i.e. the value for which $F(p) = \frac{1}{2}$), guarantees at least 50% of the optimal revenue (in expectation).

(d) Show that, under the same conditions as (c), the optimal revenue is upper bounded by the median of F.

This problem set will be discussed in the tutorials on May 4th, 2018.